

Exploring Discontinuities on a Function Graph

Informally, a function has a point of *discontinuity* at a location where its graph has a hole or break. The formal language of limits allows us to describe more precisely how the function behaves near such a location. Here is a glossary of some of the most important kinds of discontinuities for a function $y = f(x)$, described graphically and using the language of limits.

Vertical asymptote

A vertical line having equation $x = a$, such that the function values $f(x)$ increase or decrease without bound as x gets closer and closer to the value a on either side of a .

Using limit notation, the line $x = a$ is a vertical asymptote for the graph of $y = f(x)$ whenever any of the following is true:

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

Near one or both sides of a vertical asymptote, the graph of the function will appear to either rise or fall very rapidly.

Removable discontinuity (hole)

If $\lim_{x \rightarrow a} f(x) = c$ but $f(a) \neq c$ or $f(a)$ does not exist at all, then f is said to have a removable discontinuity at $x = a$.

Graphically, there is a “hole” in the graph of $y = f(x)$ at (a, c) . If we defined (or redefined) this single function value to be $f(a) = c$, then the discontinuity would be removed – hence, the name.

Essential discontinuity

If $\lim_{x \rightarrow a} f(x)$ does not exist, then we say that f has an *essential discontinuity* at $x = a$.

For example, if $x = a$ is a vertical asymptote, then the discontinuity is essential.

Another example of a discontinuity would be a *jump discontinuity*, where the left- and right-hand limits of $f(x)$ both exist as x approaches a , but the one-sided limit values do not match.

Vertical asymptotes and jumps are not the only kinds of essential discontinuities a function could have, but they do represent the most commonly encountered ones.

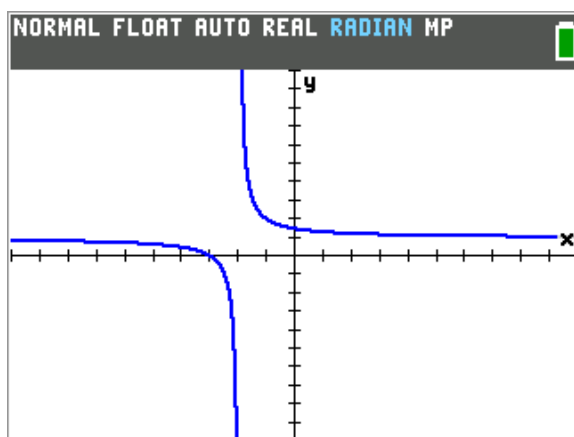
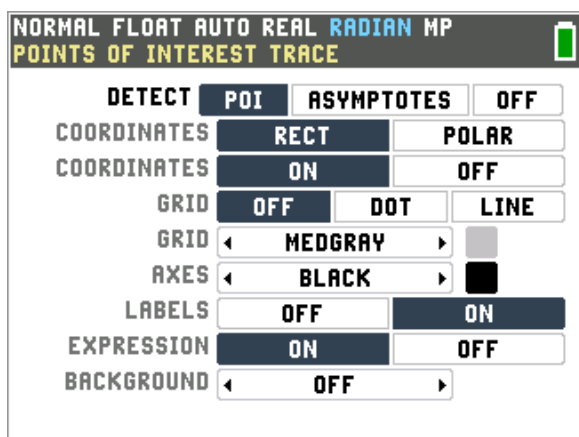
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Detecting discontinuities on the TI-84 Evo

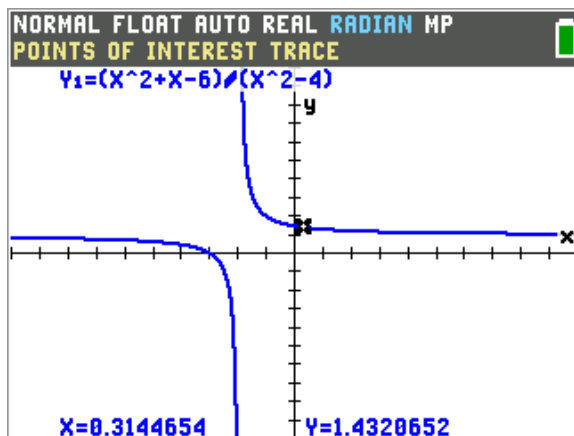
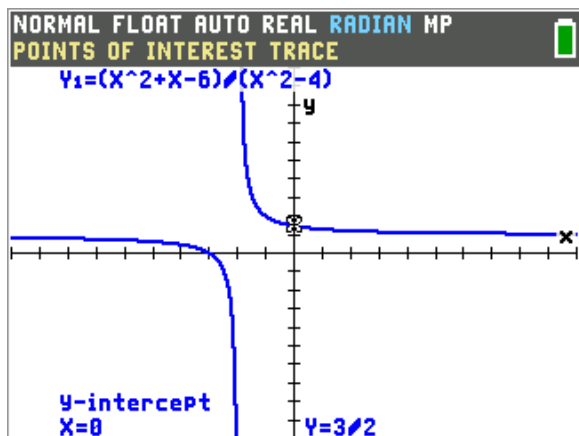
The TI-84 Evo has a function graphing format setting that attempts to detect possible essential discontinuities for a function so that the graph on screen does not connect across such a break. Here are some examples that illustrate how this feature works.

Example 1.

In $\boxed{Y=}$ enter $y = \frac{x^2+x-6}{x^2-4}$ as Y_1 . Then under $\boxed{format}$ ($\boxed{2nd}$ $\boxed{zoom}$), check that **POI** (Points of Interest) trace feature is selected. Graph Y_1 using ZStandard ($\boxed{zoom}$ $\boxed{6}$) with window settings of $Xmin = -10$, $Xmax = 10$, $Ymin = -10$, $Ymax = 10$.



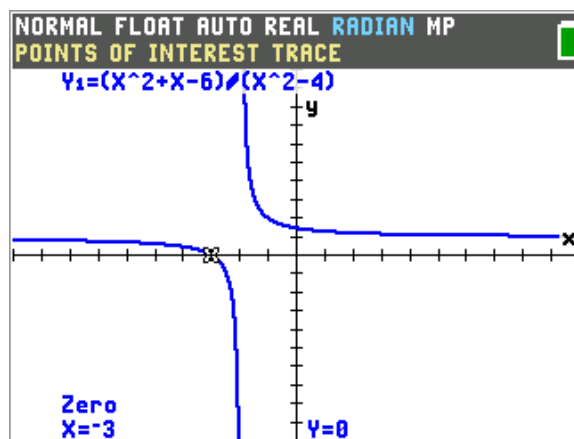
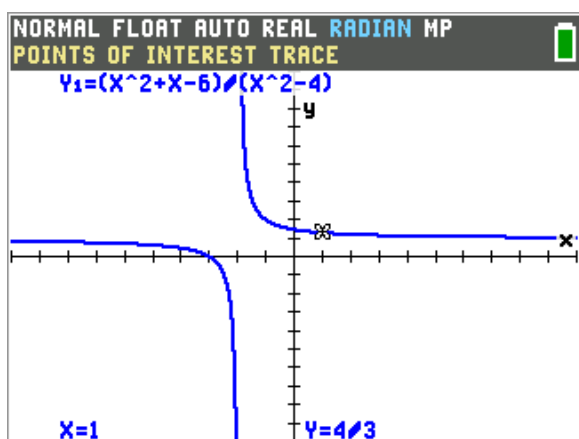
Press $\boxed{trace}$. In the ZStandard window, the starting trace values are $x = 0$ and $y = 3/2$. Note that this point is identified as a **y-intercept**. If you move the trace cursor, you will notice that the x values are not “nice” decimals.



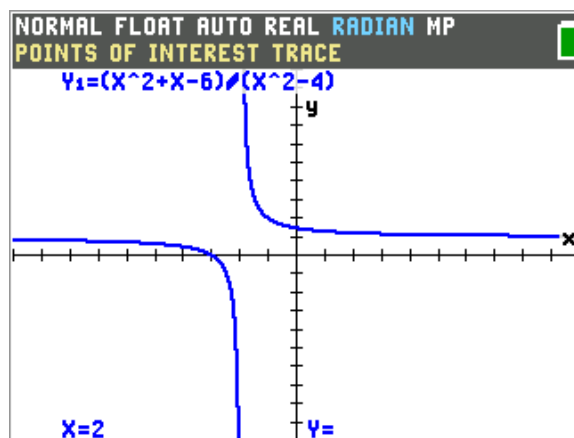
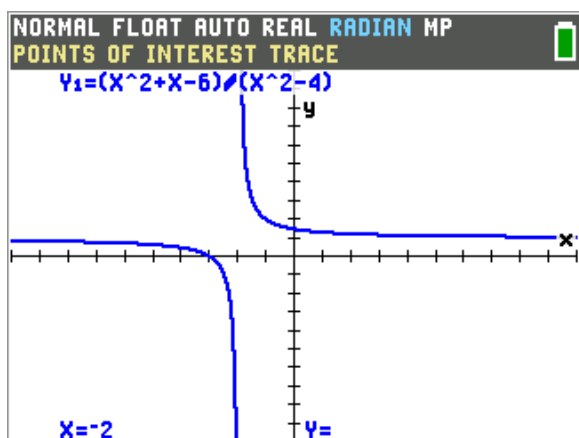
However, while tracing you can jump to a particular x value by simply typing it in.

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Try typing in the value **1** for x in trace mode and we see the corresponding y value $4/3$. Then try $x = -3$ and note that the POI trace feature identifies this as a **zero** for Y_1 .



The denominator of the expression for Y_1 is $x^2 - 4$, and has zeros at $x = -2$ and $x = 2$, so the y value will be undefined for these two values, and thus Y_1 will have discontinuities there. Type in each of these values to see that there is no trace value for y at $x = -2$ and $x = 2$.



There is an essential discontinuity at $x = -2$ and the line $x = -2$ is a vertical asymptote for the graph. At $x = 2$, we have a removable discontinuity. Notice that for each of these values, the trace cursor has disappeared, since there is no corresponding point on the function graph. No hole shows up in the graph at $x = 2$, because in the Zstandard window, the value $x = 2$ falls in between two adjacent graphed points, and the graph appears connected.

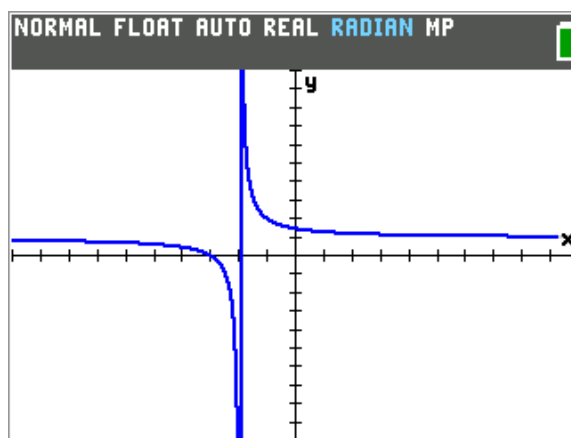
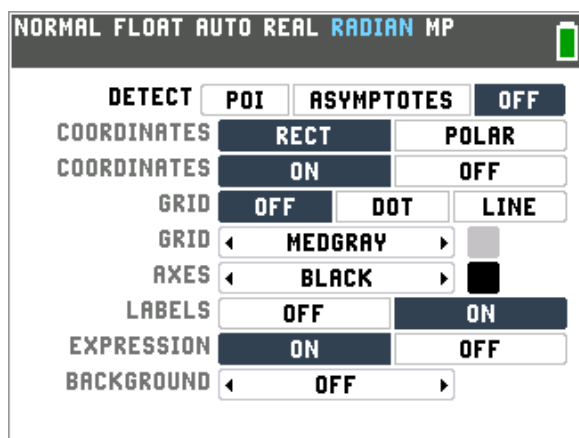
In the ZStandard window the value $x = -2$ also falls in between two adjacent graphed points, but notice that the graph is "broken" there—the graph does not connect across the vertical asymptote. With the POI trace feature on, the calculator is sensitive to the large sudden change in y value between the two adjacent x values on either side of $X = -2$, and suppresses connecting the points. (We could say that the calculator is "suspicious" that there might be a

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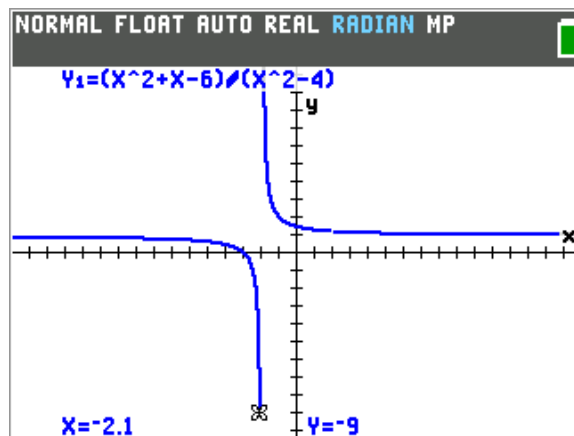
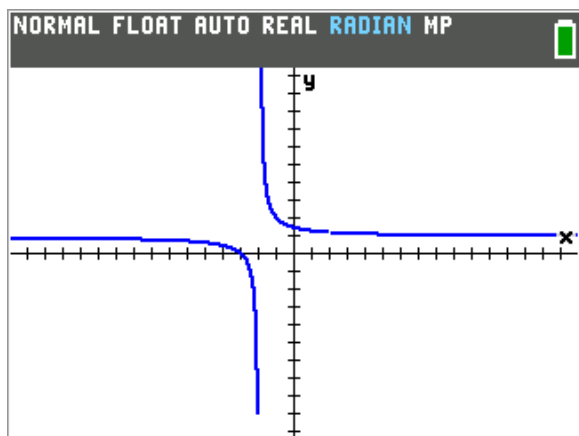
discontinuity between those two adjacent values and chooses not to connect the points since that could make the graph misleading.)

Return to [format] ($\boxed{2\text{nd}}$ $\boxed{\text{zoom}}$) and note the other two graphing options on the POI line. If the Detect **ASYMPTOTES** feature is selected, then the calculator will not identify the Points of Interest (zeros, the y-intercepts, maximums, minimums, intersection points), but it will remain sensitive to sudden changes in y values when tracing and not connect the graph when it detects these.

As you can guess, selecting **OFF** turns off detection of both points of interest and asymptotes, and if we graph Y_1 in the ZStandard window now, the graph will now show as if it were connected across the vertical asymptote location.



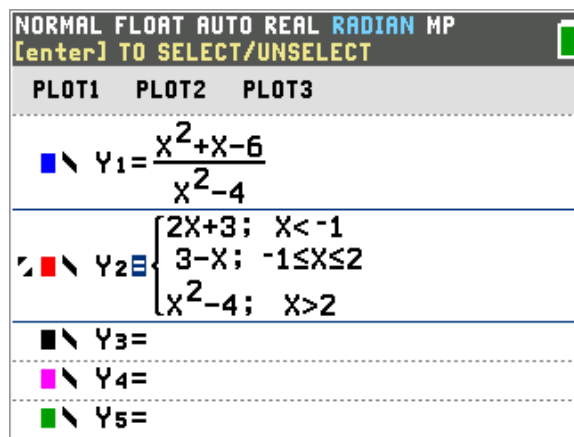
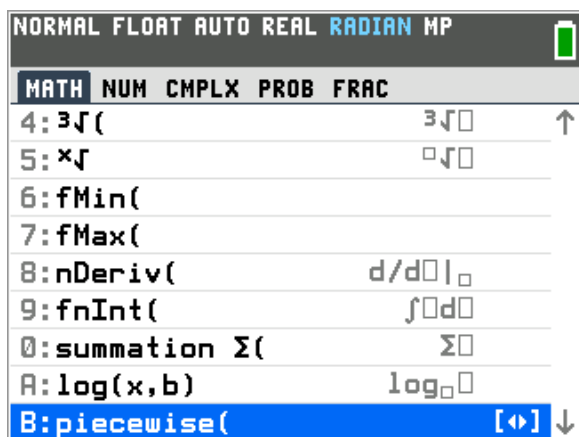
Is there any way of showing the hole in the graph? As long as we choose window settings so that a discontinuity falls exactly at a traced x value, then the calculator will not plot any point there. The screens below show the function Y_1 graphed using ZDecimal (this creates a window where the tracing will exactly hit all the integer values of x between -6 and 6 , and the trace cursor advances by exactly 0.1 in both directions). In this window, the hole appears at $x = 2$. Also, in this window, the graph will not connect across $x = -2$, even if detection of **POI** and **ASYMPTOTES** are turned off, because $x = -2$ falls exactly on a trace value.



Example 2. Piecewise defined functions and jump discontinuities

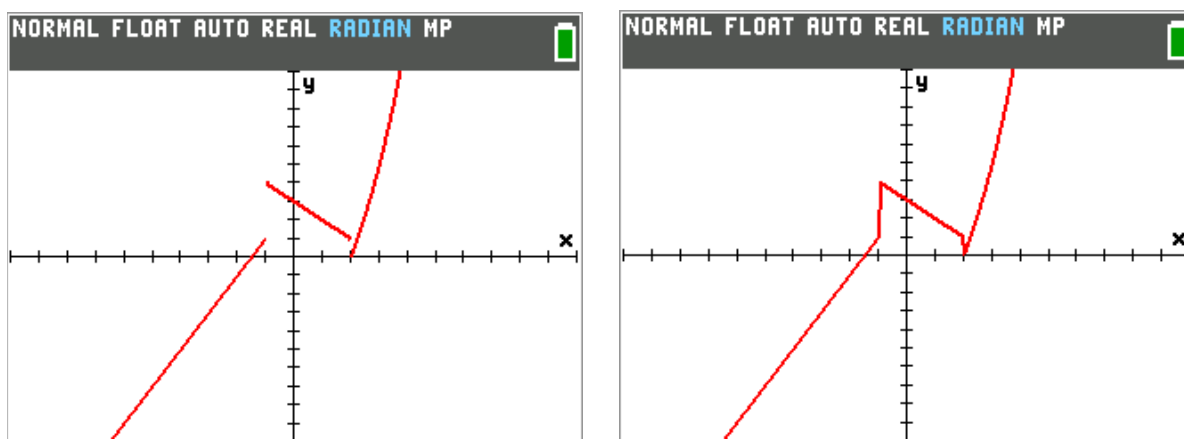
A jump discontinuity occurs at $x = a$ when the two one-sided limits both exist as x approaches a , but the values of the two limits don't match. This results in a break in the graph where the values of the function suddenly jump at $x = a$ (the gap being the difference between the two one-sided limit values there). This kind of discontinuity is essential, since it can't be fixed by simply defining or redefining the function at $x = a$.

The most common examples of functions with jump discontinuities are those with piecewise definitions— where there are different expressions for the function formula over different intervals making up pieces of the domain. To define a piecewise function in $\boxed{Y=}$ on the TI-84 Evo, there is a nice template that can be found by scrolling down in the $\boxed{\text{math}}$ menu (see left screen below). Pressing $\boxed{\text{enter}}$ on **piecewise** will bring up a window where you can specify the number of pieces the function formula has. The screen on the right below shows an example of a piecewise function defined in three pieces (the inequality symbols used to specify the different x intervals of the domain can be found in $\boxed{2\text{nd}}$ $\boxed{\text{math}}$).



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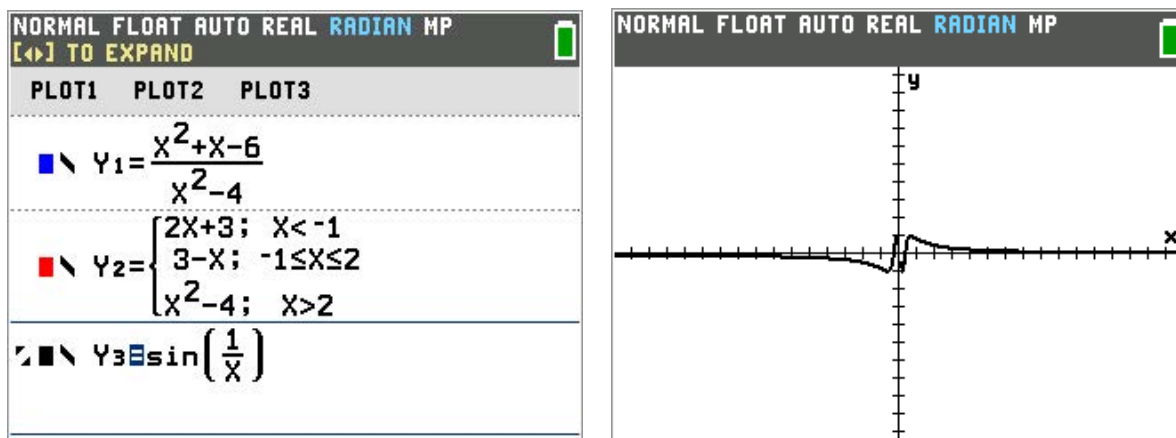
If either the **POI** or **ASYMPTOTES** detection trace feature is turned on, then the graph will not connect across a jump discontinuity. Here are two graphs of Y_2 in a ZStandard viewing window, one with the feature activated and the other with the feature turned **OFF**.



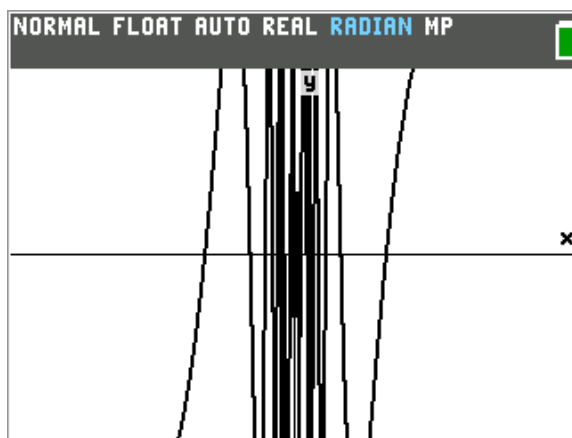
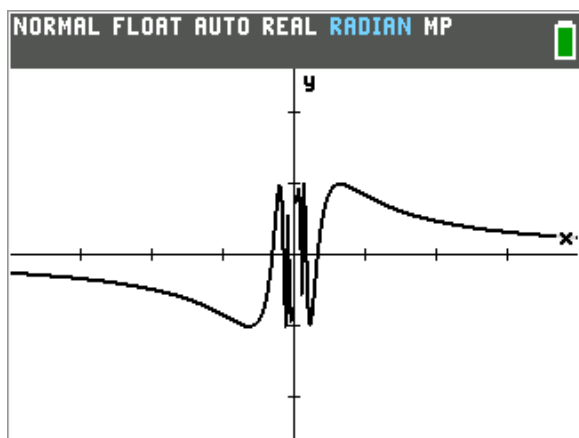
Example 3. Another discontinuity (a “wild oscillator”)

Holes, vertical asymptotes, and jumps are the most common kinds of discontinuities one encounters but there are other possibilities.

Consider the function $y = \sin\left(\frac{1}{x}\right)$. Enter this function in Y_3 and graph in a ZDecimal window.



The graph appears fairly “tame,” though we know it is undefined at $x = 0$. Indeed, one might guess that there is possibly a removable discontinuity (hole) at $x = 0$. However, if we **Zoom In** (**zoom** **2**) at the origin repeatedly, a closer look at the function’s behavior near 0 reveals wilder and wilder oscillations, with Y_3 running through all values from -1 to 1 over shorter and shorter x intervals near $x = 0$. This function has no limit as x approaches 0, so the discontinuity there is essential.



Problem 1

For each of the following functions, graph in a ZDecimal window and determine if the function has a discontinuity at the given value $x = a$. Classify each discontinuity as removable (a “hole”) or essential. If essential, classify the discontinuity as the location of vertical asymptote, a jump discontinuity, or “other”.

a) $y = \frac{x^2 - 2x - 8}{x^2 - 5x + 4}; x = 4$

b) $y = \frac{x^2 - 2x - 8}{x^2 - 5x + 4}; x = 1$

c) $y = \frac{x^2 - 2x - 8}{x^2 - 5x + 4}; x = -2$

d) $y = \frac{|x-3|}{x^2-9}; x = 3$

e) $y = \frac{|x-3|}{x^2-9}; x = -3$

f) $y = \cos \frac{3}{2-x}; x = 2$

g) $y = 2 \tan^{-1} \frac{1}{x+4}$

h) $y = \frac{1-\cos x}{x^2}; x = 0$

i) $y = \frac{\tan x}{2x-\pi}; x = \pi/2$

Problem 2

Use the Transformation Graphing App (located on the  screen) to enter $Y_1 = (X - A) * (X - B) / ((X - C) * (X - D))$.

Determine values for A, B, C, D to give the function Y_1 the following characteristics:

a) zeros $x = -2$ and at $x = 2$ and vertical asymptotes $x = -3$ and $x = 3$

- b)** zero $x = 1$, a hole at $x = 2$, and a vertical asymptote at $x = 4$
- c)** no zeros, no vertical asymptotes, and a hole at $x = -2$ and $x = 4$
- d)** a zero $x = -4$, no holes, and a single vertical asymptote $x = 3$

Problem 3

The function f is defined as $f(x) = 5 - ax$ for $x < 2$ and $f(x) = \cos(\pi x) + a$ for $x > 2$.

Find a value a that makes f continuous at all real values.